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NOTE

LOWER BOUNDS FOR BI-COLORED QUATERNARY RAMSEY NUMBERS

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In this note, we prove that $R(5, 5; 4) \geqslant 19$. We also compute lower bounds for some higher order numbers.

1. Introduction

Let S be a set of cardinality n and if r is a positive integer then $S^{(r)}$ is the family of all r-subsets of S. $S^{(r)} = (R, B)$ denotes partition of $S^{(r)}$ into two classes called "Red" and "Blue". If $p, q \geqslant r$ then for sufficiently large n every partition of $S^{(r)}$ into two classes has a p-set, all of whose r-subsets are red (red $K_p^{(r)}$) or a q-set, all of whose r-subsets are blue (blue $K_q^{(r)}$). Minimal such n is the Ramsey number $\mathrm{R}(p, q; r)$. A partition without a red $K_p^{(r)}$ and blue $K_q^{(r)}$ is called $(p, q; r)$-Ramsey partition ($(p, q; r)$-Ramsey coloring). Though F.P. Ramsey [12] proved the existence of these numbers, their computation is found to be difficult. The most widely studied numbers are bi-colored binary numbers ($r = 2$) (see [2–4, 6–8]). The case $r = 3$ has been studied in [1, 5, 9–11], but only bounds are known even for the first nontrivial case ($13 \leqslant \mathrm{R}(4, 4; 3) \leqslant 15$). In this note, we compute lower bounds for bi-colored quaternary Ramsey numbers ($\mathrm{R}(p, q; 4)$).

2. R(5, 5; 4)

Here we give a $(5, 5; 4)$-Ramsey partition on an 18-set showing $\mathrm{R}(5, 5; 4) \geqslant 19$.

Theorem 1. $\mathrm{R}(5, 5; 4) \geqslant 19$.

Proof. Let F be GF(17), Galois field on 17 elements, and let $S = F \cup \{\infty\}$. We know that the set of all transformations

$f: S \to S$ given by

$$f(x) = \begin{cases} \dfrac{ax+b}{cx+d} & \text{if } ad-bc \neq 0, \quad cx+d \neq 0 \\ \infty & \text{if } x = -\dfrac{d}{c} \\ \dfrac{a}{c} & \text{if } x = \infty \end{cases}$$

form a group under composition, call it $L(F)$. Now we partition $S^{(4)}$ into (R, B) by partitioning equivalence classes of $S^{(4)}$ under the induced action of $L(F)$. f in $L(F)$ given by

$$f(x) = \frac{x-b}{x-c} \cdot \frac{a-c}{a-b}$$

maps a, b, c onto $1, 0, \infty$ respectively. Thus, using the above transformation any quadruple $\{a, b, c, d\}$ can be mapped onto $\{1, 0, \infty, d'\}$, which we call a quadruple in standard form (denoted by $\overline{d'}$). The three orbits of $S^{(4)}$ are the following (where only standard quadruples are listed):

$$C = \{\bar{3}, \bar{6}, \bar{8}, \overline{10}, \overline{12}, \overline{15}\},$$
$$D = \{\bar{4}, \bar{5}, \bar{7}, \overline{11}, \overline{13}, \overline{14}\},$$
$$E = \{\bar{2}, \bar{9}, \overline{16}\}.$$

Now we claim that $B = C \cup E$ and $R = D$ form a $(5, 5; 4)$-Ramsey partition. To verify the claim, we need to check that every standard quintuple $\{1, 0, \infty, x, y\}$ where $\bar{x}, \bar{y} \in R$ or $\bar{x}, \bar{y} \in B$ have at least one 4-subset on either part. Further, it is enough to check this for quadruples $\{1, 0, \infty, 3, x\}$, $\bar{x} \in B$, $\{1, 0, \infty, 2, y\}$, $\bar{y} \in E$ and $\{1, 0, \infty, 4, z\}$, $\bar{z} \in R$ because there exists a transformation which sends $\{1, 0, \infty, x\}$ to $\{1, 0, \infty, 3\}$ for all $\bar{x} \in C$ and, therefore, $\{1, 0, \infty, x, y\}$ is equivalent to $\{1, 0, \infty, 3, y'\}$. Details are left to the readers. □

3. Lower bounds for higher order numbers

Abbott and Williams [1] and Kalbfleisch [11] computed lower bounds for some higher order ternary numbers by constructing Ramsey partitions using such partitions for lower order numbers as "building blocks". We prove

Theorem 2. *If* $n \leqslant \mathrm{R}(p, q-1; 4)$ *and* $m \leqslant \mathrm{R}(p-1, q; 4) - 1$, *then* $\mathrm{R}(p, q; 4) > n + m - 1$.

Proof. Proof is analogous to a similar theorem of Kalbfleisch [10] for $r = 3$. □

Table 1

p	q	$n(<\mathrm{R}(p,q;4))$
5	5	18
5	6	24
5	7	29
5	8	36
5	9	44
6	6	46
6	7	75
6	8	111
6	9	154
7	7	4096
7	8	4207
7	9	4361
8	8	8414
8	9	12775
9	9	2^{47}

Corollary. $\mathrm{R}(p,q;4)>12\binom{p+q-10}{p-5}+\binom{p+q-6}{p-3}$.

Proof. Applying Theorem 2 repeatedly (using induction) and using initial conditions $\mathrm{R}(p,4;4)=p$ and $\mathrm{R}(4,q;4)=q$, we get

$$\mathrm{R}(p,q;4)>(\mathrm{R}(5,5;4)-7)\binom{p+q-10}{p-3}+\binom{p+q-6}{p-3}. \quad \square$$

We must point out, however, that Erdős and Hajnál (see [6], 4.7) found a lower bound for $\mathrm{R}(k,k;l)$ for $k\geqslant l\geqslant 3$, which in the case of $l=4$ reduces to $\mathrm{R}(k,k;4)>2^{2^{ck^2}}$, where c is an absolute constant. Thus, the bounds given above are certainly not the best possible asymptotically. Their bound is based on the so called 'Stepping-Up Lemma'.

Lemma. *If* $\mathrm{R}(k,k;l)>n$ *and* $k\geqslant 3$ *then* $\mathrm{R}(2k+l-4,2k+l-4;l+1)>2^n$.

Using Theorem 2, its corollary, Stepping-Up Lemma and lower bounds for $\mathrm{R}(4,4;3)\geqslant 13$ [9] and $\mathrm{R}(5,4;3)\geqslant 24$ [10], we list lower bounds for some higher order numbers in *Table 1*.

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References

[1] H.L. Abbott and J.R. Williams, Lower bounds for some Ramsey numbers, J. Combin. Theory Ser. A 16 (1974) 12–17.

[2] F.R.K. Chung and C.M. Grinstead, A survey of bounds for classical Ramsey numbers, J. Graph Theory 7 (1983) 25–37.

[3] P. Erdős, Some remarks on the theory of graphs, Bull. Amer. Math. Soc. 53 (1947) 292–294.

[4] P. Erdős, Graph theory and probability II, Canad. J. Math. 13 (1961) 346–352.

[5] G, Giraud, Marjoration de nombre de Ramsey ternaire bi-colore, C.R. Acad. Sci Paris, Ser. A 269 (1969) 620–622.

[6] R. Graham, B. Rothschild and J. Spencer, Ramsey Theory (Wiley-Interscience Series, 1980).

[7] J.E. Graver and J. Yackel, Some graph theoretic results associated with Ramsey's theorem, J. Combin. Theory 4 (1968) 125–175.

[8] R.E. Greenwood and A.M. Gleason, Combinatorial relations and chromatic graphs, Canad. J. Math. 7 (1955) 1–7.

[9] J.R. Isbell, $N(4,4;3) \geq 13$, J. Combin. Theory 6 (1969) 210.

[10] J.R. Isbell, $N(5,4;3) \geq 24$, J. Combin. Theory Ser. A 34 (1983) 379–380.

[11] J.G. Kalbfleisch, On Robillard's bounds for Ramsey numbers, Canad. Math. Bull. 14 (1971) 437–449.

[12] F.P. Ramsey, On a problem in formal logic, Proc. London Math. Soc. 15 (1930) 264–286.